Matrix of projection on a plane

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Abstract

We derive the general form of the matrix of a projection from a point onto an arbitrary plane. In particular, this encompass perspective projections on plane $z = a$ and off-axis persective projection.

1 Notations and conventions

Points are noted with upper case. Geometric vectors are noted with boldface lowercase. Vectors of coordinates (with 3 or 4 coordinates) are noted with brackets and are considered row vectors. To indicate the 3 coordinates of a point or a vector, we will use the bracket. In other words:

$$[P] \equiv [P_x P_y P_z]$$
$$[\mathbf{n}] \equiv [\mathbf{n}_x \mathbf{n}_y \mathbf{n}_z]$$

To indicate a vector of 4 coordinates obtained with the 3 coordinates of a point or a vector and an extra coordinate, we will use :

$$[P \quad a] \equiv [P_z P_y P_z a]$$
$$[\mathbf{b} \quad a] \equiv [\mathbf{n}_x \mathbf{n}_y \mathbf{n}_z a]$$

Because we consider row vectors of coordinates, the multiplication of a vector by a matrix is done on the left, that is $P' = PM$ or visually :

$$\begin{array}{cc} & \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \\ \begin{bmatrix} x' & y' & z' & w' \end{bmatrix} = & \begin{bmatrix} x & y & z & w \end{bmatrix} \end{array}$$

This is disturbing for former french students like me as we learn matrix multiplication on the right (with column vectors). However, the row order for the matrix is the one used in many libraries, and in particular OpenGL.

2 Details

We are looking for the 4×4 matrix of a projection from point C on the plane of normal vector $\mathbf{n}$ and containing point P. Let's consider a point M of homogeneous coordinates $[xyzw]$ with $w \neq 0$. We note its non-homogeneous coordinates :

$$[M] = [XYZ] = [\frac{x}{w}\frac{y}{w}\frac{z}{w}]$$

Let's first assume it has a projection on the plane (that is the line (CM) is not parallel to the plane). We note this projection M' and its non-homogeneous coordinates $[X'Y'Z']$. The point M' is on line (CM) so there exists $\alpha \in \mathbb{R}$ such that :

$$\mathbf{CM'} = \alpha \mathbf{CM}$$

Since the point M' is on the plane, we have $\mathbf{PM'} \perp \mathbf{n}$, that is :

$$\begin{aligned} 0 &= \mathbf{PM'}.\mathbf{n} \\ &= (\mathbf{PC} + \mathbf{CM'}).\mathbf{n} \\ &= \mathbf{PC}.\mathbf{n} + \mathbf{CM'}.\mathbf{n} \\ &= -\mathbf{CP}.\mathbf{n} + \alpha \mathbf{CM}.\mathbf{n} \end{aligned}$$

Since by hypothesis $\mathbf{CM}.\mathbf{n} \neq 0$, it comes :

$$\alpha = \frac{\mathbf{CP}.\mathbf{n}}{\mathbf{CM}.\mathbf{n}}$$

From there, we can get the non-homogeneous coordinates of M' :

$$\begin{aligned} \mathbf{OM'} &= \mathbf{OC} + \mathbf{CM'} \\ &= \mathbf{OC} + \alpha \mathbf{CM} \\ &= \mathbf{OC} + \frac{\mathbf{CP}.\mathbf{n}}{\mathbf{CM}.\mathbf{n}} \mathbf{CM} \end{aligned}$$

We define :

$$k \equiv \mathbf{CM}.\mathbf{n}$$

and we now have :

$$\begin{aligned} k\mathbf{OM'} &= (\mathbf{CM}.\mathbf{n})\mathbf{OC} + (\mathbf{CP}.\mathbf{n})\mathbf{CM} \\ &= (\mathbf{CO}.\mathbf{n})\mathbf{OC} + (\mathbf{OM}.\mathbf{n})\mathbf{OC} + (\mathbf{CP}.\mathbf{n})\mathbf{CO} + (\mathbf{CP}.\mathbf{n})\mathbf{OM} \\ &= [(\mathbf{OC} + \mathbf{CP}).n]\, \mathbf{CO} + (\mathbf{OM}.\mathbf{n})\mathbf{OC} + (\mathbf{CP}.\mathbf{n})\mathbf{OM} \\ &= (\mathbf{OP}.n)\mathbf{CO} + (\mathbf{OM}.\mathbf{n})\mathbf{OC} + (\mathbf{CP}.\mathbf{n})\mathbf{OM} \end{aligned}$$

The above expression gives us the coordinates of M' :

$$\begin{aligned} kX' &= -(OP.n)C_x + (\mathbf{n}_x X + \mathbf{n}_y Y + \mathbf{n}_z Z)C_x + (\mathbf{CP}.\mathbf{n})X \\ &= [\mathbf{n}_x C_x + (\mathbf{CP}.\mathbf{n})]X + (\mathbf{n}_y C_x)Y + (\mathbf{n}_z C_x)Z - (OP.n)C_x \\ &= [\mathbf{n}_x C_x + (\mathbf{CP}.\mathbf{n})]\frac{x}{w} + (\mathbf{n}_y C_x)\frac{y}{w} + (\mathbf{n}_z C_x)\frac{z}{w} - (OP.n)C_x \end{aligned}$$

From which it comes (doing the same for the other coordinates) :

$$X' = \frac{1}{kw}([\mathbf{n}_x C_x + (\mathbf{CP}.\mathbf{n})]x + (\mathbf{n}_y C_x)y + (\mathbf{n}_z C_x)z - (OP.n)C_x)$$
$$Y' = \frac{1}{kw}((\mathbf{n}_x C_y)x + [\mathbf{n}_y C_y + (\mathbf{CP}.\mathbf{n})]y + (\mathbf{n}_z C_y)z - (OP.n)C_y)$$
$$Z' = \frac{1}{kw}((\mathbf{n}_x C_z)x + (\mathbf{n}_y C_z)y + [\mathbf{n}_z C_z + (\mathbf{CP}.\mathbf{n})]z - (OP.n)C_z)$$

Those are the non-homogenous coordinates of M'. From there, we can see that homogenous coordinates of M' are given by :

$$\begin{aligned}
x' &= [\mathbf{n}_x C_x + (\mathbf{CP}.\mathbf{n})]x + (\mathbf{n}_y C_x)y + (\mathbf{n}_z C_x)z - (OP.n)C_x \\
y' &= (\mathbf{n}_x C_y)x + [\mathbf{n}_y C_y + (\mathbf{CP}.\mathbf{n})]y + (\mathbf{n}_z C_y)z - (OP.n)C_y \\
z' &= (\mathbf{n}_x C_z)x + (\mathbf{n}_y C_z)y + [\mathbf{n}_z C_z + (\mathbf{CP}.\mathbf{n})]z - (OP.n)C_z \\
w' &= kw \\
&= (\mathbf{CM}.\mathbf{n})w \\
&= (\mathbf{OM}.\mathbf{n})w + (\mathbf{CO}.\mathbf{n})w \\
&= (\mathbf{n}_x X + \mathbf{n}_y Y + \mathbf{n}_z Z)w - (\mathbf{OC}.\mathbf{n})w \\
&= (\mathbf{n}_x \frac{x}{w} + \mathbf{n}_y \frac{y}{w} + \mathbf{n}_z \frac{z}{w})w - (\mathbf{OC}.\mathbf{n})w \\
&= \mathbf{n}_x x + \mathbf{n}_y y + \mathbf{n}_z z + (-\mathbf{OP}.\mathbf{n} + \mathbf{CP}.\mathbf{n})w
\end{aligned}$$

From there, it comes that the homogeneous matrix that maps M to M' is :

$$\begin{pmatrix}
\mathbf{n}_x C_x + \mathbf{CP}.\mathbf{n} & \mathbf{n}_x C_y & \mathbf{n}_x C_z & \mathbf{n}_x \\
\mathbf{n}_y C_x & \mathbf{n}_y C_y + \mathbf{CP}.\mathbf{n} & \mathbf{n}_y C_z & \mathbf{n}_y \\
\mathbf{n}_z C_x & \mathbf{n}_z C_y & \mathbf{n}_z C_z + \mathbf{CP}.\mathbf{n} & \mathbf{n}_z \\
-(\mathbf{OP}.\mathbf{n})C_x & -(\mathbf{OP}.\mathbf{n})C_y & -(\mathbf{OP}.\mathbf{n})C_z & -(\mathbf{OP}.\mathbf{n}) + \mathbf{CP}.\mathbf{n}
\end{pmatrix}$$

which can be rewritten in the very synthetic form :

$$[\mathbf{n} \quad (-\mathbf{OP}.\mathbf{n})]^T \times [C \quad 1] + (\mathbf{CP}.\mathbf{n})I$$

We know that if the plane is given by its equation $ax + by + cz + d = 0$ then we have $[\mathbf{n}] = [abc]$ and $\mathbf{OP}.\mathbf{n} = -d$. From there it comes :

$$\mathbf{CP}.\mathbf{n} = -(\mathbf{OC}.\mathbf{n} - \mathbf{OP}.\mathbf{n}) = -(aC_x + bC_y + cC_z + d)$$

and finally, we can state that:

Proposition 1. *The homogeneous matrix of a projection from a point of coordinates $[uvw]$ on a plane of equation $ax + by + cz + d = 0$ is given by the formula :*

$$[abcd]^T \times [pqr1] - (au + bv + cw + d)I$$